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COMPARISON OF FCT WITH OTHER NUMERICAL SCHEMES FOR THE BURGERS EQUATION

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ABSTRACT

The flux corrected transport (FCT) method is used to solve the linear and nonlinear Burgers Equations and to compare with other commonly used explicit methods such as the central difference (CD), the upwind (UW), and the predictor corrector or MacCormack (MC) schemes. The results show that the FCT scheme exhibits the best behavior for the linear as well as nonlinear cases. The amount of numerical diffusion is the least for the FCT scheme compared to the other schemes and more importantly there are no non-physical oscillations.

Introduction

In this paper various explicit finite difference schemes are examined to solve the widely known Burgers equation introduced about forty years ago by J.M. Burgers. The equation serves as a nonlinear analog of the fluid mechanics equations because it has terms which closely duplicate the physical properties, i.e., a convective term, a diffusive term, and a time dependent term. The Burgers equation also gives an analytic frame work for a second-order theory of finite amplitude dissipative sound propagation [1-3]. In addition, it has been used in discussions of shock structures in a Navier Stokes fluid, principally by Lagerstrom, Cole and Thrilling [4].

The distinctive feature of the Burgers equation is that it is the simplest mathematical formulation of the competition between convection and diffusion. It thus offers a relatively convenient means to quickly evaluate and compare finite-difference methods which one might wish to apply for more complicated partial differential equations. Another feature of the Burgers Equation is that although it does not have a pressure gradient term it still is a good approximation of the propagation of one-dimensional disturbances.

The present paper has two major objectives. The first is to test the flux corrected transport (FCT) scheme for solving the linear and nonlinear Burgers Equations for large cell Reynolds number Re_Δ. The second objective is to compare the FCT scheme with the other commonly used explicit methods for large cell Reynolds numbers. The numerical schemes examined here are the central difference (CD), the upwind (UW), the predictor corrector or MacCormack (MC), and the flux corrected transport (FCT) schemes.

Model Equation

The model equation that has been used to perform the numerical experiments is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2} \tag{1}$$

The linear case is obtained by replacing u, the coefficient of the convective term, by a constant c. In both cases, the equation obtained is a parabolic partial differential equation. The assumptions used in these equations are discussed in [5]. One important assumption on the finite-difference approximations is the small Mach number which permits the well known explicit stability condition [6] to be based not on the speed of sound but on the maximum flow velocity. The four numerical schemes studied are described next.

Central Difference Scheme (CD)

By applying a forward-time and centered-space difference to Eq. (1) the resulting algorithm is :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \alpha\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + O[\Delta t,(\Delta x)^2] \tag{2}$$

or

$$u_i^{n+1} = A_i\, u_{i-1}^n + B_i\, u_i^n + C_i\, u_{i+1}^n \tag{3}$$

The coefficients A_i, B_i, and C_i are given by

$$A_i = r + \frac{1}{2}\frac{\Delta t}{\Delta x}u_i^n \tag{4}$$

$$B_i = 1 - 2r \tag{5}$$

$$C_i = r - \frac{1}{2}\frac{\Delta t}{\Delta x}u_i^n \tag{6}$$

Where $r = \alpha\ \Delta t/(\Delta x)^2$ and $\upsilon = u_{max}\ \Delta t/\Delta x$ with u_{max} being the maximum value of u.

The first stability condition described in [6] requires that:

$$\nu \le 1 \tag{7}$$

This condition is required for all the schemes tested here. In addition, the central difference method requires

$$Re_\Delta \le \frac{2}{\nu} \tag{8}$$

where the mesh Reynolds number Re_Δ is defined as

$$Re_\Delta = u_{max}\frac{\Delta x}{\alpha} \tag{9}$$

For mesh Reynolds number slightly above $2/\upsilon$ the oscillations will cause the solution to "blow up" as expected from the stability analysis.

The algorithm for the linear case differs from the non-linear one in that u, the coefficient of the convective term, is replaced by c with the stability condition being the same as in the non-linear case. In both cases the resulting algorithm is explicit and first order in time.

Upwind Scheme (UW)

The upwind method uses backward differences for the convection term of Eq. (1). The numerical algorithm is given by Eq. (3), with the coefficients as

$$A_i = r + \frac{\Delta t}{\Delta x}u_i^n \tag{10}$$

$$B_i = 1 - \frac{\Delta t}{\Delta x}u_i^n - 2r \tag{11}$$

$$C_i = r \tag{12}$$

For the linear case, once again u is replaced by c. The only stability condition for the linear as well as non-linear cases is the well-known CFL condition given by Eq. (7). Note that there is no mesh Reynolds number restriction for the upwind schemes.

MacCormack Scheme (MC)

The MacCormack finite difference scheme [7] consists of two explicit steps:
Predictor step:

$$\overline{u_i^{n+1}} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_{i+1}^n - u_i^n) + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{13}$$

Corrector step:

$$u_i^{n+1} = \frac{1}{2}[\, u_i^n + \overline{u_i^{n+1}} - \frac{\Delta t}{\Delta x} \overline{u_i^{n+1}} (\overline{u_i^{n+1}} - \overline{u_{i-1}^{n+1}}) + r (\overline{u_{i+1}^{n+1}} - 2\overline{u_i^{n+1}} + \overline{u_{i-1}^{n+1}}) \,] \tag{14}$$

Here the barred quantities are evaluated in the predictor step. The corrector equation provides the final value of u at the same time level n + 1. It has been shown [7] that the method does not have any cell Reynolds number restriction and is second order accurate in both time and space. again u is replaced by c for the linear case.

Flux-Corrected Transport Scheme (FCT)

Equation (1) is solved using the flux-corrected transport algorithm [8] consisting of the following four sequential stages:

1. Compute the transported and diffused provisional values ū

$$\overline{u}_i = u_i^n - \frac{1}{2}[\, \varepsilon_{i+\frac{1}{2}} (u_{i+1}^n + u_i^n) - \varepsilon_{i-\frac{1}{2}} (u_i^n + u_{i-1}^n) \,] + \nu_{i+\frac{1}{2}} (u_{i+1}^n - u_i^n) - \nu_{i-\frac{1}{2}} (u_i^n - u_{i-1}^n) \,] \tag{15}$$

$$= u_i^n - \frac{1}{\Delta x} [\, F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}} \,]$$

where

$$\varepsilon_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} \frac{\Delta t}{\Delta x} \tag{16}$$

and

$$\nu_{i+\frac{1}{2}} = \frac{1}{2}\left|\varepsilon_i + \frac{1}{2}\right| \quad \textit{for steep discontinuity} \tag{17}$$

$$\nu_{i+\frac{1}{2}} = \frac{1}{2}\varepsilon^2_{i+\frac{1}{2}} \quad \textit{for smooth discontinuity} \tag{18}$$

The values of variables at interface i+1/2 are averages of values at cells i+1 and i, and the values at i-1/2 are averages of values at cells i and i-1. At every cell i, the $\tilde{u}_i$ differs from the u_i as a result of the fluxes of u, denoted $F_{i\pm1/2}$. The fluxes are successively added and subtracted along the array of values u_i so that the conservation of u condition is satisfied by construction. The expressions involving $\varepsilon_{i\pm1/2}$ are called the convective fluxes.

2. Compute the raw antidiffusive fluxes

The provisional values $\tilde{u}_i$ must be strongly diffused to ensure positivity. A correction to remove this strong diffusion uses additional antidiffusion fluxes given by:

$$F^{ad}_{i+\frac{1}{2}} = \mu_{i+\frac{1}{2}}\,(\overline{u_{i+1}} - \overline{u_i}) \tag{19}$$

where

$$\mu_{i+\frac{1}{2}} \approx \nu_{i+\frac{1}{2}} - \frac{1}{2}\left|\varepsilon_{i+\frac{1}{2}}\right| \quad \textit{guarantee positivity} \tag{20}$$

Antidiffusion reduces the strong diffusion, but also reintroduces the possibility of negative values or nonphysical overshoots in the new profile.

3. Correct or limit these fluxes

To obtain a positivity-preserving algorithm, the antidiffusive fluxes are modified by a process called flux correction where the corrected fluxes satisfy:

$$F^{c}_{i+\frac{1}{2}} = S.\max\{\,0, \min\{\,S.(\,\overline{u_{i+2}} - \overline{u_{i+1}}\,)\,\}, \left|F^{ad}_{i+\frac{1}{2}}\right|, S.(\,\overline{u_i} - \overline{u_{i-1}}\,)\,\} \tag{21}$$

to assure monotonicity. Here

$$S \equiv sign\,(\,\overline{u_{i+1}} - \overline{u_i}\,) \tag{22}$$

The flux correction stage should not generate new maxima or minima in the solution, nor accentuate already existing extrema.

4. Perform the final antidiffusive correction

$$u_i^n = \overline{u}_i - (F^c_{i+\frac{1}{2}} - F^c_{i-\frac{1}{2}}) \tag{23}$$

Stages 3 and 4 are the new components introduced by FCT. There are many modifications of this prescription that accentuate various properties of the solution. Some of these are summarized in [9].

Numerical Test Parameters

The test problem employed to compare the performance of the various schemes is a wave defined as follow:

$0 \le x \le 0.5 \qquad u = u_{max}$

$0.5 < x \le 1.0 \qquad u = 4.u_{max}(1-x)x$

$1.0 < x \le 5.0 \qquad u = 0.0$

For all the methods discussed in this paper $\Delta x = 0.025$, $\alpha = 0.175$, 0.0875 for all the values of Re_Δ used. A summary of the test parameters used are tabulated in Table 1.

TABLE 1
Numerical Test Parameters.

Re_Δ	1.43	3.57	4.29	10.0	28.57	57.14
Δx	0.025	0.025	0.025	0.025	0.025	0.025
Δt	0.001	0.0004	0.0005	0.0002	0.0002	0.0001
α	0.175	0.175	0.175	0.175	0.0875	0.0875
c	10.0	25.0	30.0	70.0	100.0	200.0

Results

Linear Case

Results for the linear Burgers equation are depicted in Figs. 1-3. In general, the solution exhibits a wavelike behavior with wave speed given by the constant coefficient of the convective term in the linear Burgers equation. The moving profile becomes smoother with time due to the effect of either real diffusion (the term on the right side of Eq. (1)) or numerical diffusion, the latter arising as a result of discretization. As indicated in Fig. 1, the UW scheme has the highest amount of numerical diffusion. The more important observation from the figures is that the central difference (CD) scheme, as expected, becomes unstable as the cell Reynolds number Re_Δ exceeds a value of 2.0. The comparison of Figs. 1b and 1c indicates that the oscillations leading to instability in the CD solution appear earlier in time as Re_Δ is increased. The other schemes give stable results at all Re_Δ values, with the UW method being more diffusive.

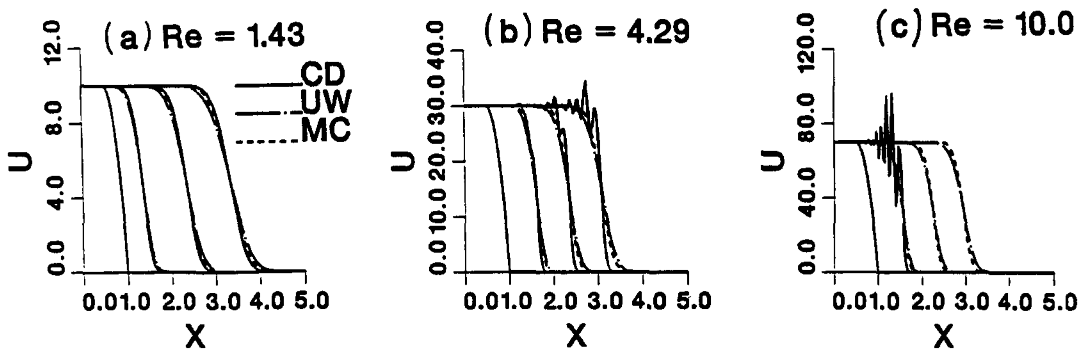

FIG. 1
Results of Numerical Experiments with Linear Burgers Equation

The comparison of MC and FCT methods is shown in Fig. 2. The solutions are almost the same except that the MC method exhibits more numerical diffusivity. In other words, the FCT scheme is better able to preserve the initial profile than the other explicit schemes studied here. From Fig. 3, similar observations can be made as in Fig. 2 except that the MC scheme shows some oscillations that become more pronounced, Table 2, as Re_Δ is increased. The profiles obtained from the FCT method, however, exhibit a smooth behavior with no sign of oscillations.

Non-Linear Case

The comparison of numerical methods for the non-linear Burgers equation is illustrated in Figs. 4-7. For the cell Reynolds number less than 2.0, Fig. 4, all the four methods yield very similar results except for the different amount of numerical diffusion for each method.

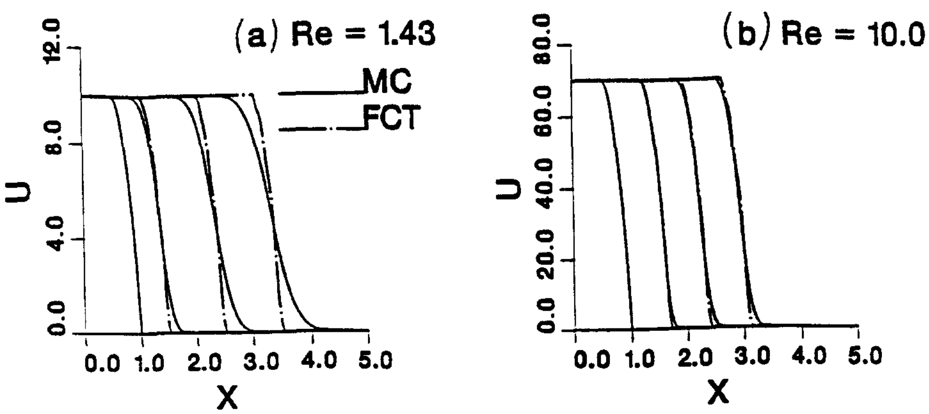

FIG. 2
Comparison of MacCormack and FCT Schemes for Linear Burgers Equation

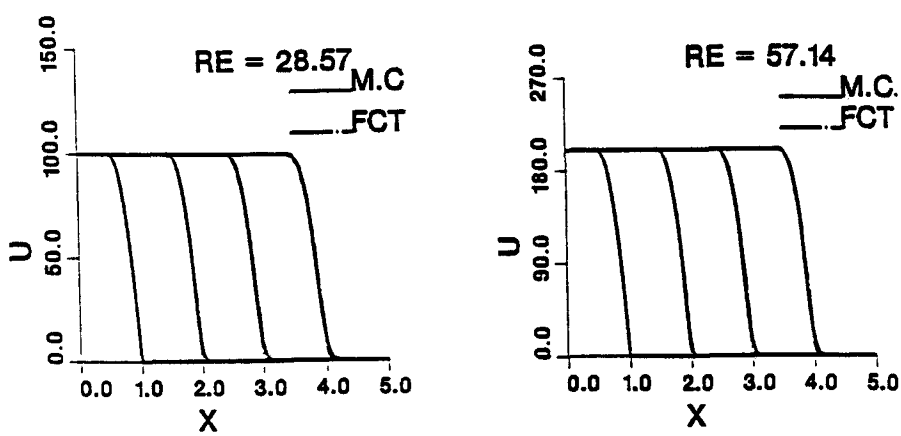

FIG. 3
Comparison of MacCormack and FCT Schemes for Linear
Burgers Equation for High Cell Reynolds Number

Again, the UW and FCT schemes exhibit the largest and smallest amounts of numerical diffusion respectively. In addition, the FCT scheme is able to preserve the initial profile better than the other three schemes. It is also noteworthy to mention that unlike the linear case the numerical diffusion, in addition to making the profile smoother, affects the wave speed for the non-linear case. For example, for the UW scheme which has the highest numerical diffusion, the wave speed is the lowest.

TABLE 2
Numerical Fluctuations for MacCormack and FCT Schemes

	$Re_\Delta = 28.57$		$Re_\Delta = 57.14$	
X	MC	FCT	MC	FCT
0.00	100.00	100.00	200.00	200.00
0.025	100.00	100.00	200.00	200.00
0.05	100.00	100.00	200.00	200.00
0.075	100.00	100.00	199.99	200.00
0.100	100.00	100.00	199.99	200.00
0.125	100.00	100.00	199.99	200.00
0.150	99.999	99.999	199.98	200.00
0.175	99.999	99.999	199.97	200.00
2.275	99.972	99.999	199.95	200.00
2.300	99.977	99.980	199.91	200.00
2.325	100.00	99.980	199.88	200.00
2.350	100.05	99.970	199.90	200.00
2.375	100.12	99.920	199.98	200.00
2.400	100.19	99.880	200.16	200.00
2.425	100.23	99.870	200.40	199.80
2.450	100.19	99.540	200.64	199.80
2.475	100.01	99.180	200.75	199.60
2.500	99.617	99.090	200.61	199.30
2.525	98.943	98.600	200.61	198.70

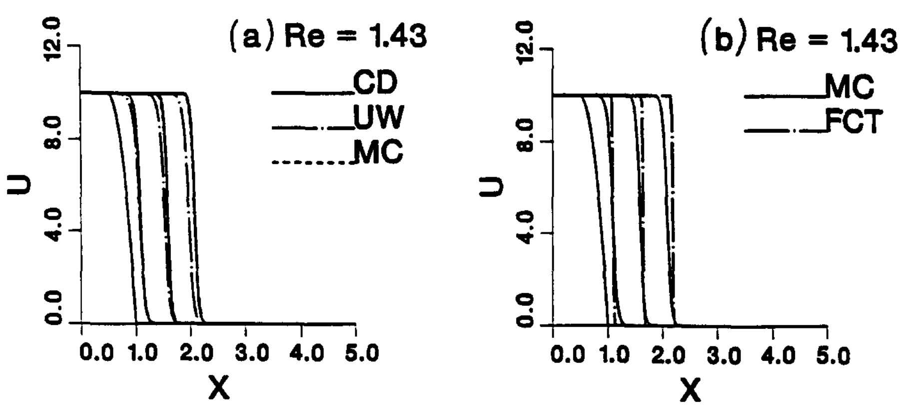

FIG. 4
Results of Numerical Experiments with Nonlinear Burgers Equation results, $Re_\Delta = 1.43$

A more interesting comparison of the methods is observed for Re_Δ greater than 2.0, as indicated in Figs. 5-7. As expected, at higher cell Reynolds numbers, the UW scheme is always stable but much more diffusive. On the other hand, the CD scheme becomes unstable almost immediately, specially for Re_Δ = 10.0. The comparison of MC and FCT schemes, given in Figs. 6-7, clearly demonstrates the superiority of FCT scheme. Not only the MC scheme has higher numerical diffusion, it also displays oscillatory behavior. Moreover, the magnitude of oscillations seems to increase as Re_Δ is increased. Although the oscillations in the MC scheme do not grow in time to make the scheme unstable, they cannot be permitted in more realistic situations; for example in the simulation of a propagating flame or a detonation. The FCT scheme, on the other hand yields a stable and almost diffusion-free solution.

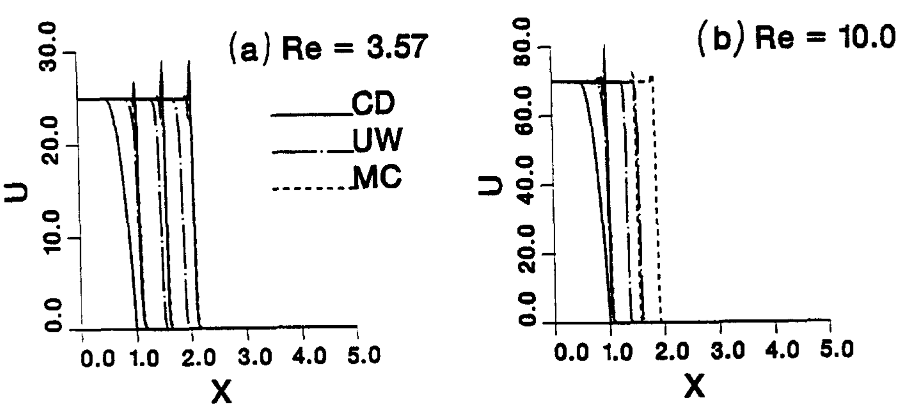

FIG. 5
Results of Numerical Experiments for Nonlinear Burgers Equation, $Re_\Delta > 2$

Concluding Remarks

Four numerical schemes have been evaluated for solving the linear and non-linear Burgers equations. The parameter varied is the mesh Reynolds number. The schemes are the upwind, the central difference, the MacCormack, and the flux corrected transport. The important conclusions are:

1. For linear as well as non-linear cases and for a mesh Reynolds number not exceeding 2.0, all the four schemes give almost identical results, except that the upwind scheme shows more numerical diffusion.

2. For a cell Reynolds number greater than two, the central difference scheme, as

expected, is unstable for linear as well as non-linear Burgers equations. The oscillations are more

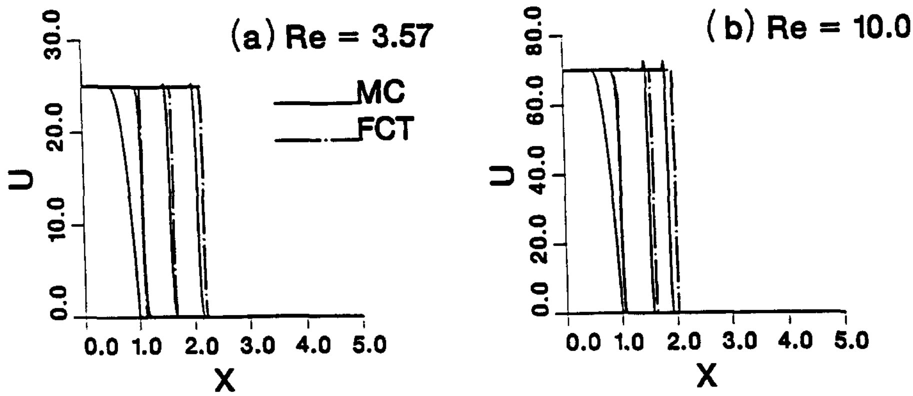

FIG. 6
Comparison of MacCormack and FCT Schemes for Nonlinear Burgers Equation, $2 < Re_\Delta \leq 10$

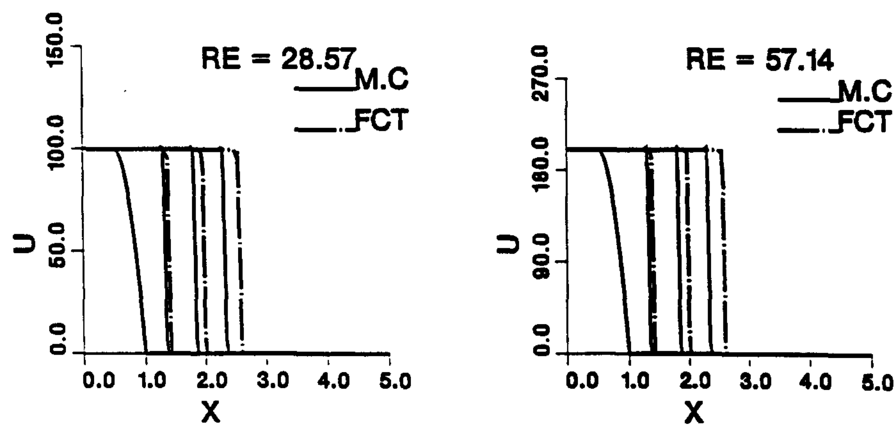

FIG. 7
Comparison of MacCormack and FCT Schemes for Nonlinear Burgers Equation, $Re_\Delta \geq 10$

pronounced and appear earlier in time for the non-linear case. The upwind scheme always yields a stable solution but at the same time exhibits a significant amount of numerical diffusion. The amount of numerical diffusion increases as Re_Δ is increased. The effect of this diffusion is to make the profiles smoother for the linear case. In the non-linear case, it also changes the effective wave speed.

3. Perhaps a more interesting result is that for Re_Δ greater than 2.0, the MacCormack scheme does not show any oscillations for the linear case. However, for the non-linear case,

numerical oscillations appear and become more pronounced as cell Reynolds number increases. Clearly, these oscillations are related to the non-linearity of the equation.

4. The FCT scheme exhibits the best behavior for the linear as well as non-linear cases. Not only the amount of numerical diffusion is the least compared to the other three schemes, there are also no non-physical oscillations. The use of this scheme is, therefore recommended for more realistic numerical simulations.

Acknowledgment

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Nomenclature

c	wave speed
F	flux
N	number of points
Re	Reynolds number
t	time
u	velocity
u_m	average velocity
α	diffusion coefficient
Δt	time step-size
Δx	spatial step-size
ε	non-dimensional numerical diffusion coefficient
υ	non-dimensional numerical diffusion coefficient

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